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# Dyonic non-Abelian vortices

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## Abstract

We study three-dimensional Yang–Mills–Higgs theories with and without a Chern–Simons interaction. We find that these theories admit a rich spectrum of vortex solitons carrying both a topological charge and a global flavour charge. We further derive a low-energy description of the vortex dynamics from a gauged linear sigma model on the vortex worldline.

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## 1. Introduction

Chern–Simons interactions have played an important role in  $d = 2 + 1$  dimensional field theories ever since they were introduced [1]. The discovery of vortices in these theories carrying both electric charge and magnetic flux has been of particular interest [2–4]<sup>1</sup>. These objects have applications in planar condensed matter physics, especially the fractional quantum Hall effect [7–9]. Further condensed matter applications of Chern–Simons interactions are considered in [10, 11]; the topic is reviewed in [12, 13].

Chern–Simons vortices in non-Abelian gauge theories have also been studied in several contexts: the nonrelativistic  $U(1)$  model of [14] was extended to  $SU(N)$  theories in [15]; vortices in a relativistic theory with a Chern–Simons term but no Yang–Mills term were considered in [16]; and the vortices of relativistic Yang–Mills–Chern–Simons (YMCS) theories were considered in [17–19].

Recently, a different type of vortex in  $U(N)$  gauge theories has been considered [20, 21]. In contrast to  $Z_N$  vortices in  $SU(N)$  theories [22, 23], bound states of these vortices are stable against decay. Moreover, these vortices have been shown to carry genuine non-Abelian flux, which appears as non-Abelian orientational zero modes. These objects have been studied mainly in  $d = 3 + 1$  dimensions, where they have been useful in understanding the quantum dynamics of gauge theories.

<sup>1</sup> Note that it is also possible to give electric charge to non-Chern–Simons vortices by coupling them to fermions [5]; this method has recently been used in studies of graphene [6].

In this paper, we extend the study of non-Abelian vortex dynamics in  $d = 2 + 1$  dimensions by considering theories with a Chern–Simons interaction which include massive matter in the fundamental representation. We find that such theories admit a rich spectrum of vortices. As well as carrying a topological (magnetic) charge and an electric charge like other Chern–Simons vortices, the vortices in our model carry a Noether charge associated with a global flavour symmetry. Unlike the electric charge, this last charge can be nonzero even if the Chern–Simons coefficient is switched off. The possibility of such dyonic vortices in non-Chern–Simons theories has previously been established in [24], but here we go further by considering the effect of the Chern–Simons interaction and studying the low-energy dynamics. We will find that the dyonic vortices in our model can follow an intricate variety of trajectories on moduli space, even when only one vortex is present.

There is a further motivation for our investigation. Besides giving us a fuller understanding of solitons in classical YMCS theory, our results on dyonic vortices help to lay the groundwork for a semiclassical quantization of the theory and an understanding of the quantum spectrum. This is interesting in light of recent research that shows that vortices in  $d = 3 + 1$  capture much of the quantum information of the bulk theory they live in [25, 26]. The question of whether analogous information can be extracted from vortices in  $d = 2 + 1$  remains open, but the work presented here is a necessary first step.

The plan of this paper is as follows. In section 2 we introduce the model of interest, then find the Bogomolnyi bound on the energy of vortex configurations and show for the first time that the dyonic vortex mass is different in the cases with and without a Chern–Simons interaction. In section 3, we review how motion on the vortex moduli space is affected by massive matter and Chern–Simons interactions, and we rederive our vortex mass results from the perspective of a  $d = 0 + 1$  gauged linear sigma model. Finally, in section 4, we look at the implications of our results in the case of a single vortex in a  $U(2)$  gauge theory. We find that there is a variety of possible BPS and non-BPS solutions for the moduli space motion, including circular orbits and more exotic looping trajectories.

## 2. Dyonic vortices in $d = 2 + 1$

We will begin by introducing the theory we are interested in, and highlighting some related models. Then we will write down the energy functional for our theory and show that it is subject to a Bogomolnyi bound. This bound gives the mass of a vortex. We will see that the mass depends both on the topological charge and on a Noether charge associated with the global flavour symmetry of the theory. The form of this dependence is different in the cases with and without a Chern–Simons interaction.

We will work with a Yang–Mills theory in  $d = 2 + 1$  dimensions, with  $\mathcal{N} = 2$  supersymmetry (i.e. four supercharges), a  $U(N)$  gauge multiplet and  $N_f$  fundamental chiral multiplets. Our theory is thus related to the dimensional reduction of the  $d = 3 + 1$ ,  $\mathcal{N} = 1$  theory in chapter 3 of [27]. However, our Lagrangian also includes two extra deformations which are allowed in  $d = 2 + 1$ ,  $\mathcal{N} = 2$  theories. The first is a Chern–Simons term, the  $\mathcal{N} = 2$  version of which is exhibited in [28]. The second is a set of real masses  $m_i$  (with  $i = 1, \dots, N_f$ ) for the chiral multiplets, as discussed in [29]. The bosonic part of the Lagrangian is then determined by supersymmetry:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2e^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{4\pi} \text{Tr} \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \frac{1}{e^2} \text{Tr} (\mathcal{D}_\mu \phi)^2 \\ & + |\mathcal{D}_\mu q_i|^2 - \sum_i q_i^\dagger (\phi - m_i)^2 q_i - \frac{e^2}{4} \text{Tr} \left( q_i q_i^\dagger - \frac{\kappa \phi}{2\pi} - v^2 \right)^2. \end{aligned} \quad (2.1)$$

Here, the adjoint scalar  $\phi$  is part of the gauge multiplet. The fundamental scalars  $q_i$  belong to the chiral multiplets; the shape of the potential for these scalars is crucial in allowing the existence of vortices. For  $N \geq 2$ , the Chern–Simons coefficient  $\kappa$  must be an integer so that the partition function is invariant under large gauge transformations. For the Abelian  $N = 1$  theory, there is no such constraint. Generically, the masses  $m_i$  break the flavour symmetry of the model from  $SU(N_f)$  to  $U(1)^{N_f-1}$ .

This model is part of a large family of theories admitting vortex solutions. We can obtain various related theories by taking limits of the above Lagrangian.

- When  $\kappa = 0$  and  $m_i = 0$ , we have a Yang–Mills theory with massless fundamental scalar fields. This theory admits non-Abelian vortices. These were originally introduced in [20, 21] and have since been studied in some detail (for reviews, see [27, 30, 31]).
- When  $\kappa = 0$  but  $m_i \neq 0$ , we have a Yang–Mills theory with massive fundamental scalar fields. The vortices of the  $d = 3 + 1$  version of this theory were discussed in [25, 32].
- When  $\kappa \neq 0$  but  $m_i = 0$ , we have a Yang–Mills–Chern–Simons–Higgs theory like those studied in [33, 34]. This reduces to the Maxwell–Chern–Simons–Higgs theory of [35, 36] in the  $N = 1$  case.
- When  $\kappa \neq 0$ ,  $m_i = 0$  and  $e^2 \rightarrow \infty$ , the Yang–Mills term vanishes, and we can integrate out  $\phi$  to get the Chern–Simons–Higgs theory with sixth-order potential. The Abelian version of this theory was first introduced in [3, 4, 37] and its non-Abelian generalization has been studied more recently in [16, 38].

The review [13] covers many of the Chern–Simons models mentioned above. Vortices in Yang–Mills–Chern–Simons theories with several Higgs fields and no fundamental matter fields have been previously studied in [17, 18, 39].

Our theory has vacua featuring different amounts of symmetry breaking. Two of the most important are the unbroken phase and the Higgs phase. In the unbroken phase, the vacuum expectation values of the fields leave the  $U(N)$  gauge symmetry intact:

$$\text{Unbroken phase: } \langle \phi_b^a \rangle = -\frac{2\pi v^2}{\kappa} \delta_b^a, \quad \langle q_i \rangle = 0. \quad (2.2)$$

Here  $a, b = 1, \dots, N$  is the colour index. The unbroken phase exists irrespective of the number of flavours  $N_f$ . The Higgs phase, on the other hand, exists only when  $N_f \geq N$ , a condition that is needed for a cancellation in the potential between the rank  $N_f$  term  $q_i q_i^\dagger$  and the rank  $N$  term  $v^2$  (which implicitly includes an  $N \times N$  identity matrix). For simplicity, we set  $N_f = N$  from here on. In the Higgs phase, the vacuum expectation values of the fields are

$$\text{Higgs phase: } \langle \phi \rangle = \text{diag}(m_1, \dots, m_N), \quad \langle q_i^a \rangle = \delta_i^a \sqrt{v^2 + \frac{\kappa}{2\pi} m_i}. \quad (2.3)$$

The flavour and gauge symmetries of the theory are each fully broken in this vacuum, but a subgroup of their product survives:  $U(N)_{\text{gauge}} \times U(1)_{\text{flavour}}^{N-1} \rightarrow U(1)_{\text{diag}}^{N-1}$ . We note that the theory also has several ground states with partly broken gauge symmetry. For each such state, the vacuum expectation values of the fields have some diagonal entries equal to those in (2.2) and the rest equal to those in (2.3).

The Higgs phase admits topologically stable vortices, and it is this ground state that we will work with for the rest of this paper. Note that if  $\sum_i m_i = M \neq 0$ , then we may define

$$v'^2 = v^2 + \frac{\kappa M}{2\pi N}, \quad \phi' = \phi - \frac{M}{N}, \quad m'_i = m_i - \frac{M}{N},$$

so that  $\sum_i m'_i = 0$ . Substituting these rescalings into (2.1) and (2.3), we find that the Lagrangian and the vacuum expectation values have exactly the same form as before, only

with  $v^2$ ,  $\phi$  and  $m_i$  replaced by  $v'^2$ ,  $\phi'$  and  $m'_i$ . Hence we may assume without loss of generality that  $\sum_i m_i = 0$ .

By a novel application of the Bogomolnyi procedure [35, 40, 41], we will now find a lower bound on the energy of the field configurations of the theory. Requiring that this bound should be saturated will give us a set of first-order differential equations whose solutions are vortices.

We first write down the energy functional  $H$  of the theory. This is the Noether charge associated with a combination of a time translation and a gauge transformation. The gauge transformation is chosen so that the infinitesimal variation of any field under the combination is proportional to that field's covariant time derivative:

$$H = \int d^2x \left[ \frac{1}{e^2} \text{Tr} (E_\alpha^2 + B^2) + \frac{1}{e^2} \text{Tr} ((\mathcal{D}_0\phi)^2 + (\mathcal{D}_\alpha\phi)^2) + |\mathcal{D}_0q_i|^2 + |\mathcal{D}_\alpha q_i|^2 + \sum_i q_i^\dagger (\phi - m_i)^2 q_i + \frac{e^2}{4} \text{Tr} \left( q_i q_i^\dagger - \frac{\kappa\phi}{2\pi} - v^2 \right)^2 \right]. \quad (2.4)$$

Note that we define  $E_\alpha = F_{0\alpha}$  and  $B = F_{12}$ . We now follow the usual Bogomolnyi procedure, and complete the square:

$$H = \int d^2x \left[ |\mathcal{D}_\pm q_i|^2 + \text{Tr} \left( \frac{1}{e} B \pm \frac{1}{2} e \left( q_i q_i^\dagger - \frac{1}{2\pi} \kappa\phi - v^2 \right) \right)^2 + \frac{1}{e^2} \text{Tr} (\mathcal{D}_0\phi)^2 + \frac{1}{e^2} \text{Tr} (E_\alpha \mp \mathcal{D}_\alpha\phi)^2 + \sum_i |\mathcal{D}_0q_i \mp i(\phi - m_i)q_i|^2 \mp \text{Tr} \left[ 2\phi \left( -\frac{1}{4\pi} \kappa B + \frac{i}{2} [(\mathcal{D}_0q_i)q_i^\dagger - q_i(\mathcal{D}_0q_i)^\dagger] + \frac{1}{e^2} \mathcal{D}_\alpha E_\alpha + \frac{i}{e^2} [\mathcal{D}_0\phi, \phi] \right) \right] \pm (\text{Tr} B)v^2 \pm i \sum_i [(q_i^\dagger(\mathcal{D}_0q_i) - (\mathcal{D}_0q_i)^\dagger q_i)m_i] \right]. \quad (2.5)$$

In this expression we have used the notation  $\mathcal{D}_\pm = \mathcal{D}_1 \pm i\mathcal{D}_2$ .

If  $\kappa$  and  $m_i$  are chosen to be zero then we may set  $\phi \equiv 0$ , so the third line in (2.5) disappears. However, if either  $\kappa$  or  $m_i$  is nonzero and we have nontrivial fields  $q_i$ ,  $A_\alpha$ , then the equation of motion for  $\phi$  includes source terms, so we cannot set  $\phi \equiv 0$ . Thus for the third line in (2.5) to disappear, we must invoke Gauss' law:

$$-\frac{\kappa B}{4\pi} + \frac{i}{2} [(\mathcal{D}_0q_i)q_i^\dagger - q_i(\mathcal{D}_0q_i)^\dagger] + \frac{1}{e^2} \mathcal{D}_\alpha E_\alpha + \frac{i}{e^2} [\mathcal{D}_0\phi, \phi] = 0. \quad (2.6)$$

Of the two terms in the last line of (2.5), the first integrates to  $\pm 2\pi k v^2$ , where  $k$  is the topological charge. The integral of the second term can be written as  $\pm \sum_i Q_i m_i$ , where we define

$$Q_i \equiv i \int d^2x (q_i^\dagger(\mathcal{D}_0q_i) - (\mathcal{D}_0q_i)^\dagger q_i). \quad (2.7)$$

The  $Q_i$  are conserved Noether charges associated with the  $U(1)^{N-1}$  flavour symmetry of the theory. If we take the trace of Gauss' law (2.6) and then integrate over space, we find that  $\sum_i Q_i = \kappa k$ . This is the realization in our model of the usual relationship between 'electric' charge and magnetic flux for Chern–Simons vortices. In the  $e^2 \rightarrow \infty$  limit where the Yang–Mills term decouples, the local version of the relationship is given by  $2\pi \sum_i \rho_i = \kappa \text{Tr} B$ , where  $\rho_i$  is the integrand in (2.7).

All of the terms in the first two lines of (2.5) are non-negative squares, so we see that the energy  $H$  of our configuration is subject to a Bogomolnyi bound, which we interpret as the vortex mass:

$$H \geq \left| 2\pi k v^2 + \sum_i Q_i m_i \right| \equiv M_{\text{vortex}}. \quad (2.8)$$

This new result is remarkable because it means that sometimes increasing the value of  $|\sum_i Q_i m_i|$  may decrease the vortex mass. This is an unusual property; typically, increasing the size of a charge on a soliton will increase the soliton's mass.

To saturate the bound (2.8), a configuration must obey Gauss' law (2.6) and solve the first-order Bogomolnyi equations that arise from setting the squared terms to zero in (2.5). Given a configuration that satisfies these constraints, it is straightforward to show that the quantity  $2\pi k v^2 + \sum_i Q_i m_i$  is strictly positive if  $k > 0$  and strictly negative if  $k < 0$ . For the case  $k = 1$ , we will show in the next section that the minimum possible value of  $M_{\text{vortex}}$  for given  $m_i$  is  $2\pi v^2 + m_p \kappa$ , where  $m_p$  is the most negative of the  $m_i$  values if  $\kappa > 0$  or the most positive of the  $m_i$  values if  $\kappa < 0$ .

In the special case  $\kappa = 0$ , the Bogomolnyi process works slightly differently. The absence of the  $\kappa B$  term in Gauss' law means that in (2.5) we can make one sign choice for the first line and the topological charge term, and a separate choice for the second and third lines and the flavour charge term. As a result, we have a stricter Bogomolnyi bound:

$$H \geq |2\pi k v^2| + \left| \sum_i Q_i m_i \right| \equiv M_{\text{vortex}}. \quad (2.9)$$

Once again, the bound is saturated if and only if Gauss' law and the Bogomolnyi equations hold.

We call the vortices in this theory dyonic because  $M_{\text{vortex}}$  has contributions from two types of charge: the topological charge  $k$  and the Noether charge  $\sum_i Q_i m_i$ . The significance of the difference between the bounds (2.8) and (2.9) is that increasing  $|\sum_i Q_i m_i|$  always increases the vortex mass in the  $\kappa = 0$  case, but may either increase or decrease the vortex mass in the  $\kappa \neq 0$  case.

The dyonic vortices we have considered in  $d = 2 + 1$  dimensions are related to several other kinds of dyonic solitons in different numbers of dimensions. These include dyonic instantons in  $d = 4 + 1$  [42], the original dyonic monopoles in  $d = 3 + 1$  [43–45], semilocal vortices in  $d = 2 + 1$  [46]<sup>2</sup> and the dyonic domain walls known as Q-kinks in  $d = 1 + 1$  [49]. A wide variety of soliton states, including vortices and their dyonic extensions, were considered in [24, 50–52].

It is interesting to compare the dyonic vortex masses we have calculated with the masses of other types of dyons. For dyonic monopoles and Q-kinks, the mass can be written schematically as  $M = (Q_a^2 + Q_b^2)^{1/2}$ , where  $Q_a$  and  $Q_b$  are the two types of charge that make these solitons dyonic [49, 53]. We can therefore regard dyonic monopoles and Q-kinks as bound states, each containing two distinct objects with masses  $Q_a$  and  $Q_b$ . For dyonic instantons and semilocal vortices, the mass is schematically closer to  $M = Q_a + Q_b$  [42, 46], so it seems these objects are only threshold bound states.

<sup>2</sup> Semilocal vortices can arise in Abelian gauge theories with multiple flavours of matter and are related to sigma model Q-lumps. The moduli space of semilocal vortices suffers from non-normalizable zero modes corresponding to the scale parameters of the vortices [47, 48]. Our non-Abelian model avoids this problem because it contains equal numbers of colours and flavours: it hence has no vortex scale parameters, so the vortex moduli space is free of non-normalizable zero modes.

The dyonic vortices in our study, meanwhile, can be regarded as true bound states of two different types of object. The first object is a vortex with a topological charge but no flavour charges; this has mass  $2\pi|k|v^2$ . The second object has flavour charges but no topological charge, so it is made up of excitations of the squark fields  $q_i^a$ . Excitations of the  $q_i^a$  about the vacuum have masses  $[e^2\langle q_i^a \rangle^2/2 + (m_a - m_i)^2]^{1/2}$ . The dyonic vortex mass  $M_{\text{vortex}}$  is smaller than the sum of the masses of the constituent objects, since the  $\langle q_i^a \rangle$  part of the squark excitation masses does not contribute to  $M_{\text{vortex}}$ . Hence the dyonic vortex is a classically stable bound state.

### 3. The dyonic vortex worldline theory

The moduli space approximation is a useful tool for understanding the low-energy dynamics of vortices [54]. For the theory given by (2.1), with  $\kappa$  and  $m_i$  set to zero, slow motion on the vortex moduli space is described by a nonlinear sigma model. In this section, we will review how this motion is modified when we reintroduce nonzero  $m_i$  and  $\kappa$ . We will then reformulate our description of the motion as a gauged linear sigma model, and see how from this perspective we can rederive the results of the preceding section by applying the Bogomolnyi procedure to the  $d = 0 + 1$  energy functional.

In the theory (2.1) with  $m_i = 0$  and  $\kappa = 0$ , the set of static physical solutions of the Bogomolnyi equations with topological charge  $k$  forms a moduli space  $\mathcal{M}$ . This has collective coordinates  $X^p$  and Kähler metric  $g_{pq}$ , where  $p, q = 1, \dots, 2N|k|$ . Low-energy vortex dynamics is described by geodesic motion on  $\mathcal{M}$ :

$$L = \frac{1}{2}g_{pq}(X)\dot{X}^p\dot{X}^q. \quad (3.1)$$

The metric has an  $SU(N)$  isometry arising from the unbroken  $SU(N)_{\text{diag}}$  remnant of the gauge and flavour symmetries of the theory. We can find a set of  $N$  Killing vectors  $\mathcal{K}_i$  associated with the isometry such that  $\sum_i \mathcal{K}_i = 0$ .

If we allow  $m_i \neq 0$  but keep  $\kappa = 0$ , then a potential is induced on the moduli space. This potential can be written in terms of the Killing vectors  $\mathcal{K}_i$  [25, 32, 55]:

$$V_m = \sum_{i,j=1}^N (m_i \mathcal{K}_i^p)(m_j \mathcal{K}_j^q)^\dagger g_{pq}. \quad (3.2)$$

If we instead allow  $\kappa \neq 0$  but keep  $m_i = 0$ , then a magnetic field  $\mathcal{F} \in \Omega^2(\mathcal{M})$  is induced on the moduli space. Writing this locally as  $\mathcal{F} = d\mathcal{A}$  and working to leading order in  $\kappa e/v$ , we find the dynamics on  $\mathcal{M}$  is modified by an extra term in the Lagrangian [33, 36]:

$$L_\kappa = -\kappa \mathcal{A}_p(X)\dot{X}^p. \quad (3.3)$$

If we allow both  $m_i \neq 0$  and  $\kappa \neq 0$ , then the Lagrangian for motion on  $\mathcal{M}$  includes both the Chern–Simons term (3.3) and a potential similar to (3.2). We might expect that the potential (3.2) is modified by terms of order  $m_i \kappa$ ; we have not calculated this from the nonlinear sigma model, but in the next subsection we will check that it is true, at least for  $k = 1$ , by using a gauged linear sigma model.

#### 3.1. The gauged linear sigma model for one vortex

For an alternative description of motion on the one-vortex moduli space, and to clarify what happens when  $\kappa$  and  $m_i$  are both nonzero, we now assemble a standard  $d = 0 + 1$  gauged linear sigma model, with a  $U(1)$  gauge field and  $\mathcal{N} = (0, 2)$  supersymmetry. This ‘worldline theory’ for a non-Abelian vortex is straightforward to solve, but it is more interesting than

the corresponding theory for an Abelian vortex because non-Abelian vortices have an internal orientation as well as a spatial position. If we ignore the spatial position of our non-Abelian vortex and set  $m_i = 0$ , then the moduli space of vortex configurations is  $\mathcal{M} = \mathbb{C}\mathbb{P}^{N-1}$ , with homogeneous coordinates  $\varphi_i \in \mathbb{C}, i = 1, \dots, N$ . These coordinates give  $N$  fundamental scalar fields in the worldline theory, and roughly speaking they correspond to the orientation of the vortex in the  $U(N)$  gauge group of the parent theory. The fields  $\varphi_i$  are subject to the  $D$ -term constraint

$$D \equiv \sum_{i=1}^N \varphi_i \varphi_i^\dagger - r = 0, \tag{3.4}$$

where  $r$  is a constant which is determined to be  $2\pi/e^2$  [20, 21, 56]. In the case  $m_i = 0, \kappa = 0$ , the Lagrangian (3.1) is  $L = |\mathcal{D}_t \varphi_i|^2$ , where  $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - iA\varphi_i$  and  $A$  is the gauge field for the  $U(1)$  worldline gauge symmetry. The scalars  $\varphi_i$  enjoy an  $SU(N)$  flavour symmetry corresponding to the  $SU(N)$  isometry of the metric on  $\mathcal{M}$ .

If we introduce nonzero masses  $m_i$  for the  $q_i$  of the parent theory, then the gauged linear sigma model is deformed by the introduction of masses  $m_i$  for the  $\varphi_i$  [25]:

$$L = |\mathcal{D}_t \varphi_i|^2 - \sum_i \varphi_i^\dagger (\sigma - m_i)^2 \varphi_i. \tag{3.5}$$

Here,  $\sigma$  is an adjoint scalar field in the  $\mathcal{N} = (0, 2)$  gauge multiplet. The deformation (3.5) is consistent with the  $\mathcal{N} = (0, 2)$  supersymmetry of the theory, but for generic  $m_i$  it breaks the flavour symmetry from  $SU(N)$  to  $U(1)^{N-1}$ . When we integrate out  $\sigma$  and  $A$  in (3.5), we recover the moduli space potential (3.2).

If we now introduce a Chern–Simons interaction into the parent theory, then a corresponding term is induced in the gauged linear sigma model [33]:

$$L = |\mathcal{D}_t \varphi_i|^2 - \sum_i \varphi_i^\dagger (\sigma - m_i)^2 \varphi_i - \kappa(A + \sigma). \tag{3.6}$$

Although each of the terms here has been studied before, we believe this is the first time the Lagrangian including both the masses  $m_i$  and the Chern–Simons coefficient  $\kappa$  has been analysed. Note that we can again integrate out  $\sigma$  and  $A$  to find the potential on  $\mathcal{M}$ :

$$V = \frac{1}{r} \sum_{i < j} (m_i - m_j)^2 |\varphi_i|^2 |\varphi_j|^2 + \frac{\kappa}{r} \sum_i m_i |\varphi_i|^2. \tag{3.7}$$

The first term is independent of  $\kappa$  and matches the potential in [25]. The second term is the promised order  $m_i \kappa$  modification. Note that the first term is non-negative, so  $V$  can be minimized (subject to the constraint (3.4)) by choosing  $p \in 1, \dots, N$  such that  $m_p \kappa \leq m_i \kappa \forall i$ , then setting

$$|\varphi_i| = \begin{cases} r & \text{if } i = p \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

From the Lagrangian (3.6), we can compute Gauss’ law,

$$i \sum_{i=1}^N ((\mathcal{D}_t \varphi_i) \varphi_i^\dagger - \varphi_i (\mathcal{D}_t \varphi_i)^\dagger) = \kappa \tag{3.9}$$

$$\Rightarrow \sum_i \mathcal{Q}_i = \kappa, \quad \text{where } \mathcal{Q}_i \equiv i((\mathcal{D}_t \varphi_i) \varphi_i^\dagger - \varphi_i (\mathcal{D}_t \varphi_i)^\dagger). \tag{3.10}$$

The  $\mathcal{Q}_i$  are conserved Noether charges corresponding to the  $U(1)^{N-1}$  flavour symmetry of the worldline theory. As we will see soon, they are related to the Noether charges  $Q_i$  (2.7) of the parent theory.



### 3.2. The dyonic vortex mass

We are now ready to apply the Bogomolnyi procedure to rederive the new results of section 2. As in the  $d = 2 + 1$  case, we can find the energy functional for the theory as the Noether charge associated with a combined time translation and gauge transformation:

$$H_{1d} = |\mathcal{D}_t \varphi_i|^2 + \sum_i \varphi_i^\dagger (\sigma - m_i)^2 \varphi_i + \kappa \sigma. \quad (3.11)$$

When we complete the square, we get

$$H_{1d} = |\mathcal{D}_t \varphi_i - i(\sigma - m_i)\varphi_i|^2 + \text{Tr} \left[ \sigma \left( \kappa - i \sum_i ((\mathcal{D}_t \varphi_i)\varphi_i^\dagger - \varphi_i(\mathcal{D}_t \varphi_i)^\dagger) \right) \right] + \sum_i \mathcal{Q}_i m_i. \quad (3.12)$$

Imposing Gauss' law (3.9) gives the lower bound  $H_{1d} \geq \sum_i \mathcal{Q}_i m_i$ . Note that if  $m_i = 0$ , then the vortex has mass  $2\pi v^2$ . The energy  $H_{1d}$  gives corrections to this mass which arise from the fields on the worldline, so we can write

$$M_{\text{vortex}} = 2\pi v^2 + \sum_i \mathcal{Q}_i m_i. \quad (3.13)$$

This formula is remarkable because it shows from the worldline perspective that appropriately chosen values for the charges  $\mathcal{Q}_i$  can give a negative contribution to the mass of the vortex. This is an unusual property for a soliton, but it matches what we found in the parent theory in section 2.

By eliminating the auxiliary fields  $\sigma$  and  $A$  from (3.11), we can see that the minimal possible value of  $M_{\text{vortex}}$  for given  $m_i$  is achieved by setting  $\dot{\varphi}_i = 0$  and then choosing  $\varphi_i$  to satisfy (3.8), so that  $V$  is minimized. We then have  $M_{\text{vortex}} = 2\pi v^2 + m_p \kappa$ , where  $m_p$  is the most negative of the  $m_i$  values if  $\kappa > 0$  or the most positive of the  $m_i$  values if  $\kappa < 0$ . This minimal value of  $M_{\text{vortex}}$  is guaranteed to be strictly positive by the fact that the vacuum expectation value  $\langle q_p^p \rangle$  given in (2.3) must be strictly positive to permit the winding of the vortex to lie in the  $p$ th flavour, as demanded by (3.8).

In the special case  $\kappa = 0$ , the Bogomolnyi process works slightly differently, just as in three dimensions. In particular, we have a choice of signs when we complete the square:

$$H_{1d} = |\mathcal{D}_t \varphi_i \pm i(\sigma - m_i)\varphi_i|^2 \pm \text{Tr} \left[ \sigma \left( i \sum_i ((\mathcal{D}_t \varphi_i)\varphi_i^\dagger - \varphi_i(\mathcal{D}_t \varphi_i)^\dagger) \right) \right] \mp \sum_i \mathcal{Q}_i m_i. \quad (3.14)$$

This gives us a stricter bound on the energy once we apply Gauss' law:

$$H_{1d} \geq \left| \sum_i \mathcal{Q}_i m_i \right| \Rightarrow M_{\text{vortex}} = 2\pi v^2 + \left| \sum_i \mathcal{Q}_i m_i \right|. \quad (3.15)$$

Our results (3.13), (3.15) match the  $k = 1$  versions of the corresponding results (2.8), (2.9) from the parent theory if we make the identification  $\mathcal{Q}_i = Q_i$ . Note that under this identification, there is agreement between the constraints  $\sum_i Q_i = \kappa k$  and  $\sum_i \mathcal{Q}_i = \kappa$ , and the right-hand side of (3.13) is necessarily positive. Just as we saw in section 2, increasing  $|\sum_i \mathcal{Q}_i m_i|$  always increases the vortex mass in the  $\kappa = 0$  case, but may either increase or decrease the vortex mass in the  $\kappa \neq 0$  case.

If we are interested in a multi-vortex system, it is again possible to provide a gauged linear sigma model for the dynamics, this time by using a brane construction [20]. However, the metric induced on the moduli space when we quotient the gauged linear sigma model

by the action of the gauge group no longer matches the metric  $g_{pq}$  of the nonlinear sigma model. Consequently, the  $k$ -vortex gauged linear sigma model does not give a strictly accurate description of  $k$ -vortex moduli space dynamics, although it may still be useful for finding qualitative properties of the motion.

#### 4. Example: a single $U(2)$ vortex

In this section, which contains original material, we will apply our results to the case of a single vortex in a  $U(2)$  Yang–Mills–Chern–Simons theory including two flavours of matter with masses  $m$  and  $-m$ . We will obtain the effective potential on the moduli space of vortices, and use it to study some features of the dynamics. In particular, we will find that the gauged linear sigma model perspective makes it straightforward to find BPS and non-BPS solutions corresponding to several different types of moduli space motion.

If  $m = 0$ , the moduli space  $\mathcal{M}$  is  $\mathbf{S}^2 \cong \mathbf{CP}^1$ . When we switch on the mass  $m$ , a potential given by (3.7) is induced on  $\mathcal{M}$ . Using coordinates  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi)$  on  $\mathbf{S}^2$ , we have

$$L = \frac{r}{4}[\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2] + \frac{\kappa}{2}(\cos \theta - 1)\dot{\phi} - rm^2 \sin^2 \theta - \kappa m \cos \theta. \quad (4.1)$$

The first term describes a standard sigma model on  $\mathbf{S}^2$ . The second term is the Dirac monopole connection, as was explained in [33], and the final two terms give the potential induced by  $m$ . Changing the sign of  $m$  corresponds to changing which pole we identify as north. Changing the sign of  $\kappa$  corresponds to performing a parity transformation  $\theta \rightarrow \pi - \theta$ . In what follows, we choose both  $m$  and  $\kappa$  to be non-negative without loss of generality.

The theory described by (4.1) has a conserved angular momentum:

$$J = r\dot{\phi} \sin^2 \theta + \kappa \cos \theta = Q_1 - Q_2. \quad (4.2)$$

There is also a conserved Hamiltonian which we can write as

$$H = \frac{r}{4}\dot{\theta}^2 + V_{\text{eff}}(\theta), \quad \text{where} \quad V_{\text{eff}}(\theta) = \frac{(J - \kappa \cos \theta)^2}{4r \sin^2 \theta} + rm^2 \sin^2 \theta + m\kappa \cos \theta. \quad (4.3)$$

The shape of the effective potential  $V_{\text{eff}}$  determines the dynamics of the system. Writing  $c = \cos \theta$ ,  $s = \sin \theta$ , we find

$$V'_{\text{eff}}(\theta) = 2rm^2 s^{-3} \left( c^2 - \frac{\kappa}{2mr}c + \frac{J}{2mr} - 1 \right) \left( c^3 - c \left( \frac{J}{2mr} + 1 \right) + \frac{\kappa}{2mr} \right). \quad (4.4)$$

##### 4.1. The shape of $V_{\text{eff}}$ when $\kappa = 0$ and $m \neq 0$

When  $J = 0$ ,  $V_{\text{eff}}$  has a maximum at  $\theta = \pi/2$  and two minima at  $\theta = 0, \pi$ . When  $|J| > 0$ ,  $V_{\text{eff}} \rightarrow \infty$  as  $\theta \rightarrow 0$  or  $\pi$ . For  $0 < |J| < 2mr$ ,  $V_{\text{eff}}$  has a maximum at  $\theta = \pi/2$  and two minima at the values of  $\theta$  given by  $\sin \theta = \sqrt{|J|/(2mr)}$ , as shown in figure 1. For  $|J| \geq 2mr$ ,  $V_{\text{eff}}$  has a single minimum at  $\theta = \pi/2$ .

The energy of a solution sitting at a minimum of  $V_{\text{eff}}$  is given by

$$H_{1d} = \begin{cases} 2rm^2 & |J/(2mr)| & \text{for } |J| \leq 2mr \\ rm^2 & [(J/(2mr))^2 + 1] & \text{for } |J| > 2mr. \end{cases} \quad (4.5)$$

To understand this, we turn to the Bogomolnyi equations that arise from (3.14). These are satisfied by setting  $\dot{\theta} = 0$  along with either  $\theta = 0, \theta = \pi$  or  $\dot{\phi} = 2m \text{sign}(mJ)$ . For  $|J| \leq 2mr$ , solutions to the Bogomolnyi equations are precisely the configurations that sit at minima of  $V_{\text{eff}}$ ; these saturate the Bogomolnyi bound  $H_{1d} \geq |\sum_i Q_i m_i| = |mJ|$ . For  $|J| > 2mr$ , the Bogomolnyi equations cannot be solved, and the energy exceeds the Bogomolnyi bound even at the minimum of  $V_{\text{eff}}$ .

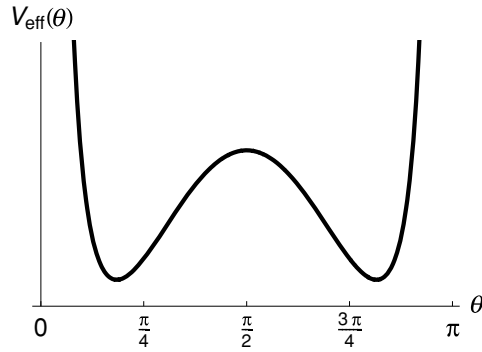


Figure 1. A sketch of the effective potential when  $\kappa = 0$  and  $0 < |J| < 2mr$ .

#### 4.2. The shape of $V_{\text{eff}}$ when $\kappa > 0$

For nonzero  $\kappa$ ,  $V_{\text{eff}}$  can take on a rich variety of different shapes. We find different patterns of behaviour in the cases  $0 < \kappa < mr/2$ ,  $mr/2 < \kappa < 4mr$  and  $4mr < \kappa$ , and in this section we will consider each of these cases in turn.

In every case, we have  $V_{\text{eff}} \rightarrow \infty$  as  $\theta \rightarrow 0$  or  $\pi$ , except where stated otherwise. The Bogomolnyi equations arising from (3.12) are solved by setting  $\dot{\theta} = 0$  and either  $\theta = 0$ ,  $\theta = \pi$  or  $\phi = 2m$ ; note that by (4.2),  $\phi$  is a function of  $\theta$ . To satisfy  $\dot{\theta} = 0$ , a configuration must sit at an extremum of  $V_{\text{eff}}$ . We will refer to a minimum of  $V_{\text{eff}}$  as ‘BPS’ if it is located at a value of  $\theta$  such that  $\theta = 0$ ,  $\theta = \pi$  or  $\phi = 2m$ . Otherwise, we will say the minimum is ‘non-BPS’. At BPS minima, the Bogomolnyi bound is saturated, so  $V_{\text{eff}} = mJ$ . At non-BPS minima,  $V_{\text{eff}} > mJ$ . We will see that minima of  $V_{\text{eff}}$  often occur at  $\theta = f_{\pm}$ , where we define

$$f_{\pm}(\kappa, J) = \cos^{-1} \left( \frac{1}{2} \left( \frac{\kappa}{2mr} \pm \sqrt{\left( \frac{\kappa}{2mr} \right)^2 - 4 \left( \frac{J}{2mr} - 1 \right)} \right) \right).$$

We first describe the shape of  $V_{\text{eff}}$  in the case  $0 < \kappa < mr/2$  for different values of  $J$ . When  $|J| < \kappa$ ,  $V_{\text{eff}}$  has a non-BPS minimum and a maximum at values of  $\theta$  less than  $\pi/2$ , and a BPS minimum at  $\theta = f_-$ . A sketch is given in figure 2. When  $\kappa \leq J < 2mr + \kappa^2/(8mr)$ ,  $V_{\text{eff}}$  has two BPS minima at  $\theta = f_{\pm}$ , with a maximum in between. A sketch of  $V_{\text{eff}}$  would then look like figure 1, but with the positions of the extrema shifted sideways; an exception is the special case  $J = \kappa$ , when one of the minima is actually at  $\theta = 0$ . When  $J > 2mr + \kappa^2/(8mr)$ ,  $V_{\text{eff}}$  has a single non-BPS minimum at a value of  $\theta$  in  $(0, \pi/2)$ .

Now suppose we decrease  $J$  until it is negative. When  $J = -\kappa$ ,  $V_{\text{eff}}$  has a BPS minimum at  $\theta = \pi$ , and a non-BPS minimum and a maximum at values of  $\theta$  in  $(0, \pi/2)$ . For  $2mr(3[\kappa/(4mr)]^{2/3} - 1) < J < -\kappa$ ,  $V_{\text{eff}}$  has a non-BPS minimum at a value of  $\theta$  in  $(\pi/2, \pi)$ , and a non-BPS minimum and a maximum at values of  $\theta$  in  $(0, \pi/2)$ . A sketch of  $V_{\text{eff}}$  would look similar to figure 2. Finally, when  $J < 2mr(3[\kappa/(4mr)]^{2/3} - 1)$ ,  $V_{\text{eff}}$  has a single non-BPS minimum at a value of  $\theta$  in  $(\pi/2, \pi)$ . This concludes our description of possible shapes of  $V_{\text{eff}}$  when  $0 < \kappa < mr/2$ .

The pattern of possible shapes is slightly different if we choose  $mr/2 < \kappa < 4mr$  because now  $-\kappa < 2mr(3[\kappa/(4mr)]^{2/3} - 1)$ . For  $-\kappa \leq J \leq 2mr(3[\kappa/(4mr)]^{2/3} - 1)$ ,  $V_{\text{eff}}$  has a single BPS minimum at  $\theta = f_-$ ; note that  $f_- = \pi$  for  $J = -\kappa$ . For  $J < -\kappa$  or  $J > 2mr(3[\kappa/(4mr)]^{2/3} - 1)$ , the behaviour matches what we had in the case  $0 < \kappa < mr/2$

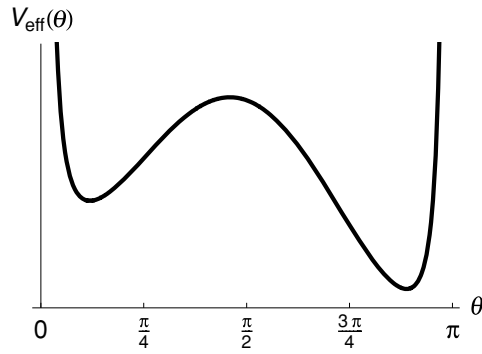


Figure 2. A sketch of the effective potential when  $0 < \kappa < mr/2$  and  $|J| < \kappa$ .

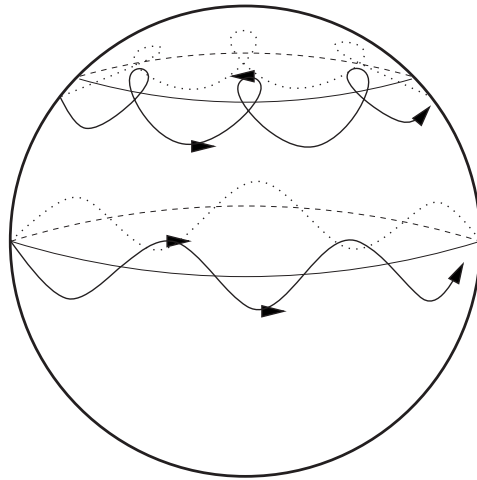


Figure 3. Possible moduli space paths with nonzero  $\kappa$  and  $m$ .

for  $J < 2mr(3[\kappa/(4mr)]^{2/3} - 1)$  or  $J > -\kappa$ , respectively. In the case  $\kappa > 4mr$ ,  $V_{\text{eff}}$  always has a single minimum. For  $|J| \leq \kappa$ , the minimum is BPS and is located at  $\theta = f_-$ . In particular, the minimum is at  $\theta = 0$  when  $J = \kappa$  and at  $\theta = \pi$  when  $J = -\kappa$ . For  $J < -\kappa$ , the minimum is non-BPS and is at a value of  $\theta$  in  $(0, \pi/2)$ ; for  $J > \kappa$ , the minimum is non-BPS and is at a value of  $\theta$  in  $(\pi/2, \pi)$ . Note that the results of this paragraph still apply in the special case  $m = 0$  if we redefine  $f_- = \cos^{-1}(J/\kappa)$ .

#### 4.3. Possible types of motion

Each minimum of  $V_{\text{eff}}$  gives a stable solution of the equations of motion of the system. Usually, such a solution is a circular orbit around  $\mathbf{S}^2$  with fixed nonzero  $\dot{\phi}$  at a fixed value of  $\theta$ . However, if the minimum occurs at  $\theta = 0$  or  $\pi$ , then it gives a static solution sitting at a pole of  $\mathbf{S}^2$ . Also, for  $m = 0$  and  $|J| < \kappa$ , the BPS minimum of  $V_{\text{eff}}$  corresponds to a static solution with  $\dot{\phi} = 0$  sitting at a value of  $\theta$  in  $(0, \pi)$ ; at this value of  $\theta$ , the effects of  $J$  and  $\kappa$  effectively cancel one another out.

If we give the system an energy greater than that required to sit at a given minimum of  $V_{\text{eff}}$ , then  $\theta$  oscillates around that minimum and the motion on  $\mathbf{S}^2$  can take two distinct forms, as shown in figure 3. For  $|J| > \kappa$ , the sign of  $\dot{\phi}$  always matches that of  $J$ . We then have motion similar to the lower, wave-like path in figure 3. Things are more interesting for  $|J| < \kappa$ . Then

$$\text{sign}(\dot{\phi}) = \text{sign}\left(\theta - \cos^{-1}\left(\frac{J}{\kappa}\right)\right).$$

If the minimum of  $V_{\text{eff}}$  that we are oscillating around is at the critical value  $\theta = \cos^{-1}(J/\kappa)$  then we always have motion similar to the upper, looping path in figure 3. If the minimum of  $V_{\text{eff}}$  that we are oscillating around is not at the critical value of  $\theta$ , then we can have a wave-like path or a looping path depending on whether there is enough energy for  $\theta$  to cross the critical value as it oscillates.

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